Department of Mathematics & Philosophy of Engineering

Faculty of Engineering Technology

The Open University of Sri Lanka

Nawala - Nugegoda

Academic Year:	2012/2013	Assignment No.	04
Course Code:	MPZ3231		

1)

(i) (a)

(b)

(ii)

(iii) (a)

(b)

2)

(a)From the Mover's theorem $\left(\cos 6\theta + i\sin 6\theta\right) = \left(\cos\theta + i\sin\theta\right)^6$

$$\cos 6\theta + i\sin 6\theta = {}^6C_0\cos^6\theta + {}^6C_1\cos^5\theta i\sin\theta + {}^6C_2\cos^4\theta i^2\sin^2\theta$$

$$+{}^6C_3\cos^3\theta i^3\sin^3\theta + {}^6C_4\cos^2\theta i^4\sin^4\theta + {}^6C_5\cos\theta i^5\sin^5\theta + {}^6C_6 i^6\sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta$$

$$-\sin^6\theta + i\left(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta\right)$$

Equating the real parts $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

$$= \cos^6\theta - 15\cos^4\theta\left(1-\cos^2\theta\right) + 15\cos^2\theta\left(1-\cos^2\theta\right)^2 - \left(1-\cos^2\theta\right)^3$$

$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta$$

$$+15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$$

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

Equating the imaginary parts

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

$\tan 6\theta = \frac{\sin 6\theta}{\cos 6\theta}$

$= \frac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{32\,cos^6\theta - 48\,\,cos^4\theta + 18\,cos^2\theta - 1}$

$= \frac{6\tan\theta - 20\tan^3\theta + 6\,\,\tan^5\theta}{32\,\,-48\,\,sec^2\theta + 18\,sec^4\theta - sec^6\theta}$

$= \frac{6\tan\theta - 20\tan^3\theta + 6\,\,\tan^5\theta}{32\,\,-48\,\,(1+tan^2\theta) + 18\,(1+tan^2\theta)^2 - (1+tan^2\theta)^3}$

$= \frac{6\tan\theta - 20\tan^3\theta + 6\,\,\tan^5\theta}{1 - 15\,tan^2\theta + 15\,tan^4\theta + tan^6\theta}$

$= \frac{2\tan\theta\,(3 - 10\tan^2\theta + 3\tan^4\theta)}{1 - 15\,tan^2\theta + 15\,tan^4\theta + tan^6\theta}$

$= \frac{2t(3 - 10\,t^2 + 3t^4)}{1 - 15\,t^2 + 15\,t^4 + t^6}$

(b) From the Mover's theorem $(\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3\cos^2\theta\,\, i\sin\theta + 3\cos\theta i^2\sin^2\theta + i^3\sin^3\theta$

$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

Equating the real parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

Equating the imaginary parts

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\,cos^2\theta\sin\theta - sin^3\theta}{cos^3\theta - 3\cos\theta\,sin^2\theta} = \frac{3\tan\theta - tan^3\theta}{1 - 3\,tan^2\theta}$$

$t = \tan\theta$: $\tan 3\theta = \frac{3\,t - t^3}{1 - 3\,t^2}$

If $\tan 3\theta = 1$, $\frac{3\,t - t^3}{1 - 3\,t^2} = 1 \quad\rightarrow\quad 1 - 3\,t^2 = 3\,t - t^3$

$$\tan 3\theta = 1 = \tan\left({}^{\pi}/_{4} \pm n\pi\right) \quad : n = 0,1,2,3, \ldots$$
$$\theta = {}^{\pi}/_{12} \pm n\,{}^{\pi}/_{3}$$
$$t = \tan\left({}^{\pi}/_{12} \pm n\,{}^{\pi}/_{3}\right) = 0.26794, -1, 3.732$$

Algebraic method:

$$1 - 3\,t^2 = 3\,t - t^3$$
$$t^3 - 3t^2 - 3t + 1 = 0$$
$$(t+1)(t^2 - 4t + 1) = 0$$
$$t = -1, \qquad t = \frac{4 \pm \sqrt{4^{2-4}}}{2} = 2 \pm \sqrt{3}$$
$$i.e.\;\; t = 0.26794, -1, 3.732$$

(c) $\cos 6\theta = 32\,cos^6\theta - 48\,\,cos^4\theta + 18\,cos^2\theta - 1$

$32\,cos^6\theta - \cos 6\theta = 48\,\,cos^4\theta - 18\,cos^2\theta + 1$

$$cos^4\theta = (cos^2\theta)^2 = \left(\frac{\cos 2\theta + 1}{2}\right)^2 = \frac{cos^2 2\theta + 2\cos 2\theta + 1}{2}$$
$$= \frac{\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1}{2} = \frac{\cos 4\,\theta + 1 + 4\,cos\,2\theta + 2}{4}$$

$$48\int cos^4\theta\, d\theta = \frac{48}{4}\int (\cos 4\,\theta + 4\,cos\,2\theta + 3)\,d\theta = 24\left(\sin 4\theta/_4 + {}^{4\sin 2\theta}/_2 + 3\theta\right)$$
$$= 6\sin 4\theta + 48\sin 2\theta + 72\,\theta \text{ --------------(1)}$$
$$cos^2\theta = \frac{\cos 2\theta + 1}{2}$$
$18\int cos^2\theta\, d\theta = \frac{18}{2}\int(\cos 2\theta + 1)d\theta = {}^9/_2 \sin 2\theta + 9\theta$------------------(2)

(1) and (2)

$$32\,cos^6\theta - \cos 6\theta = 6\sin 4\theta + 48\sin 2\theta + 72\,\theta - {}^9/_2 \sin 2\theta - 9\theta + 1$$
$$= 6\sin 4\theta + {}^{87}/_2 \sin 2\theta + 63\theta + 1$$

(d)$-1 = z^3 \quad z = \cos\theta + i\sin\theta$

$$-1 = (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$
$$\cos\big((2n+1)\pi\big) + i\,sin\,\big((2n+1)\pi\big) = \cos 3\theta + i\sin 3\theta$$
$$3\theta = (2n+1)\pi$$
$$\theta = \left(\frac{2n+1}{3}\right)\pi$$

$$n = 0: z_1 = \cos\left({}^\pi/_3\right) + i\sin\left({}^\pi/_3\right) = {}^1/_2 + i\,{}^{\sqrt{3}}/_2$$
$$n = 1: z_2 = \cos(\pi) + i\sin(\pi) = -1$$
$$n = 2: z_3 = \cos\left({}^{5\pi}/_3\right) + i\sin\left({}^{5\pi}/_3\right) = {}^1/_2 - i\,{}^{\sqrt{3}}/_2$$

Let $z_1 = \lambda = \cos\left({}^\pi/_3\right) + i\sin\left({}^\pi/_3\right) = {}^1/_2 + i\,{}^{\sqrt{3}}/_2$

$$\lambda^2 = \left(\cos\left({}^\pi/_3\right) + i\sin\left({}^\pi/_3\right)\right)^2 = \cos\left({}^{2\pi}/_3\right) + i\sin\left({}^{2\pi}/_3\right) = -\,{}^1/_2 + i\,{}^{\sqrt{3}}/_2$$
$$= -\left({}^1/_2 - i\,{}^{\sqrt{3}}/_2\right) = -(z_3)$$

$\therefore z_2 = -\lambda^2$

$\therefore$ roots are -1, $\lambda^2, -\lambda^2$

$$\lambda^2 - \lambda + 1 = -\left({}^1/_2 - i\,{}^{\sqrt{3}}/_2\right) - \left({}^1/_2 + i\,{}^{\sqrt{3}}/_2\right) + 1 = 0$$

(c))$-1 = z^7$

$$-1 = (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$$
$$\cos\big((2n+1)\pi\big) + i\,sin\,\big((2n+1)\pi\big) = \cos 7\theta + i\sin 7\theta$$
$$7\theta = (2n+1)\pi$$
$$\theta = \left(\frac{2n+1}{7}\right)\pi$$
$$n = 0: z_1 = \cos\left({}^\pi/_7\right) + i\sin\left({}^\pi/_7\right) = 0.901 + i\,0.434$$
$$n = 1: z_2 = \cos\left({}^{3\pi}/_7\right) + i\sin\left({}^{3\pi}/_7\right) = 0.223 + i\,0.975$$
$$n = 2: z_3 = \cos\left({}^{5\pi}/_7\right) + i\sin\left({}^{5\pi}/_7\right) = -0.624 - i\,0.782$$
$$n = 3: z_4 = \cos\left({}^{7\pi}/_7\right) + i\sin\left({}^{7\pi}/_7\right) = -1$$
$$n = 4: z_4 = \cos\left({}^{9\pi}/_7\right) + i\sin\left({}^{9\pi}/_7\right)$$
$$n = 5: z_5 = \cos\left({}^{11\pi}/_7\right) + i\sin\left({}^{11\pi}/_7\right)$$
$$n = 6: z_6 = \cos\left({}^{13\pi}/_7\right) + i\sin\left({}^{13\pi}/_7\right)$$

$$\cos\left({}^\pi/_7\right) + \cos\left({}^{3\pi}/_7\right) + \cos\left({}^{5\pi}/_7\right) = 0.5$$

(3)

(i)

	Male(M)	Female(F)	
Like	500	200	700
Dislike	250	350	600
Neutral	125	75	200
	875	625	1500

(a) $P(D \cap M) = {}^{250}/_{1500} = {}^{1}/_{6}$

(b) $P({}^{L}/_{F}) = \frac{P(L\cap F)}{P(F)} = \frac{{}^{200}/_{1500}}{{}^{625}/_{1500}} = 0.32$

(c) $P(D) = P({}^{D}/_{M})P(M) + P({}^{D}/_{F})P(F)$

$= \left({}^{250}/_{875} \times {}^{875}/_{1500}\right) + \left({}^{350}/_{625} \times {}^{625}/_{1500}\right) = 0.4$

$P(D \cup M) = P(D) + P(M) - P(D \cap M) = \frac{4}{10} + \left(\frac{2}{15}\right) - \frac{1}{6} = \frac{11}{30}$

(ii)

a) The probability he has only BSc. Degree = ${}^{150}/_{200}$ = 0.25

b) Pr(Student who has master degree, given that he is over 40)

$$\Pr({}^{A}/_{M}) = \frac{P(A\cap M)}{P(M)} = \frac{{}^{10}/_{200}}{{}^{50}/_{200}} = \frac{10}{50} = 0.2$$

c) Pr(under 30, given that he has only a bachelor's degree)

$$\Pr({}^{A}/_{B}) = \frac{P(A\cap B)}{P(B)} = \frac{{}^{90}/_{200}}{{}^{150}/_{200}} = \frac{90}{150} = 0.6$$

4)

F = A answers the first question

E = Game finishes after the first question

W = A wins

$\Pr({}^{W}/_{F}) = Pr(W|F \cap E)\ Pr(E|F) + Pr(W|F \cap E)\ Pr(E'|F)$ —— (1)

$Pr(E|F)$ = Pr (Student who has master degree, given that he is over 40) $= \alpha$

$Pr(E'|F) = 1 - \alpha$

$Pr(W|F \cap E) = 1$

Similarly, $Pr(W|F \cap E)$ = Pr $(W|F')$

Substitute the above values in equation (1)

Pr $(W|F)$ = (1 x α) + Pr $(W|F')$ x $(1 - \alpha) = \alpha +$ Pr $(W|F')$ x $(1 - \alpha)$ —— (2)

Further, Pr $(W|F')\ Pr(W|F' \cap E)\ Pr(E|F') + Pr(W|F' \cap E')\ Pr(E'|F')$

$Pr(E|F')$ = Pr (B answers first question correctly) $= \beta$

$Pr(E'|F') = 1 - \beta$, $Pr(W|F \cap E) = 0$

In addition, $Pr(W|F' \cap E') =$ Pr $(W|F)$

Pr $(W|F')$ = (0 x β) + Pr $(W|F)$ x $(1 - \beta)$ = Pr $(W|F)$ x $(1 - \beta)$ —— (3)

(2) and (3) Pr $(W|F) = \alpha +$ Pr $(W|F)\ (1 - \alpha)\ (1 - \beta)$

Pr $(W|F)\{ 1 -$ Pr $(W|F)\ (1 - \alpha)\ (1 - \beta)\} = \alpha$

Pr $(W|F) = {}^{\alpha}/_{1 - (1 - \alpha)(1 - \beta)}$

$$\Pr\ (W|F') = {(1-\beta)\,\alpha}\Big/{1-(1-\alpha)\,(1-\beta)}$$

5) $\quad S_0 \;\; S_1 \;\; S_2 \;\; S_3 \;\; S_4 \;\; S_5 \;\; S_6 \;\; S_7$

(i)Sample Space $S = \{000,001,010,011,100,101,110,111\}$

We need to find

$$P\left({S_i}/{R_{101}}\right); i = 0,1,2,\ldots,7$$

$$P(R_{101}) = \sum_{i=0}^{7} P\left({R_{101}}/{S_i}\right) P(s_i)$$

$$P\left({R_{101}}/{S_0}\right) = (1-\alpha)\alpha(1-\alpha) = \alpha(1-\alpha)^2$$

$$P\left({R_{101}}/{S_1}\right) = (1-\alpha)(\alpha)^2$$

$$P\left({R_{101}}/{S_2}\right) = (1-\alpha)(1-\alpha)(1-\alpha) = (1-\alpha)^3$$

$$P\left({R_{101}}/{S_3}\right) = \alpha(1-\alpha)^2$$

$$P\left({R_{101}}/{S_4}\right) = (1-\alpha)(\alpha)^2$$

$$P\left({R_{101}}/{S_5}\right) = \alpha^3$$

$$P\left({R_{101}}/{S_6}\right) = \alpha(1-\alpha)^2$$

$$P\left({R_{101}}/{S_7}\right) = (1-\alpha)(\alpha)^2$$

But we know

$$P(transmitting\ signal\ 1) = {4}/{7}$$

$P(S_0) = ({3}/{7})^3 \qquad P(S_4) = ({4}/{7})({3}/{7})^2$

$P(S_1) = ({4}/{7})({3}/{7})^2 \qquad P(S_5) = ({4}/{7})^2({3}/{7})$

$P(S_2) = ({4}/{7})({3}/{7})^2 \qquad P(S_6) = ({4}/{7})^2({3}/{7})$

$P(S_3) = ({4}/{7})^2({3}/{7}) \qquad P(S_6) = ({4}/{7})^3$

$$P(R_{101}) = \sum_{i=0}^{7} P\left({R_{101}}/{S_i}\right) P(s_i)$$

$$= \alpha(1-\alpha)^2({3}/{7})^3 + (1-\alpha)(\alpha)^2({4}/{7})({3}/{7})^2 + (1-\alpha)^3({4}/{7})({3}/{7})^2$$
$$+ \alpha(1-\alpha)^2({4}/{7})^2({3}/{7}) + (1-\alpha)(\alpha)^2({4}/{7})({3}/{7})^2 + \alpha^3({4}/{7})^2({3}/{7})$$
$$+ \alpha(1-\alpha)^2({4}/{7})^2({3}/{7}) + (1-\alpha)(\alpha)^2({4}/{7})^3$$

$P(R_{101}) = 48\alpha^2 + 136(1-\alpha)(\alpha)^2 + 123\alpha(1-\alpha)^2 + 36(1-\alpha)^3$

Now

$$P\left(S_5/_{R_{101}}\right) = \frac{P\left(R_{101}/_{S_5}\right)P(s_5)}{P(R_{101})}$$

$$= \frac{\alpha^3 \times {}^{48}/_{343}}{48\alpha^2 + 136(1-\alpha)(\alpha)^2 + 123\alpha(1-\alpha)^2 + 36(1-\alpha)^3/_{343}}$$

$$= \frac{48\,\alpha^3}{48\alpha^2 + 136(1-\alpha)(\alpha)^2 + 123\alpha(1-\alpha)^2 + 36(1-\alpha)^3}$$

(ii)

	A	B	C
	2x	x	x
defective	2%	2%	4%

The prior probabilities are same

$$P(A) = {}^2/_4\ , P(B) = {}^1/_4\ , P(C) = {}^1/_4$$

Say D is defective item

$P(D\ /\ A) = 0.02$, $P(D\ /\ B) = 0.02$ $\quad P(D\ /\ C) = 0.04$

$$P\left({}^A/_D\right) = \frac{P(D\ /\ A)P(A)}{P(A)P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{0.02 \times 1/4}{0.02 \times 2/4 + 0.02 \times 1/4 + 0.04 \times 1/4} = 0.4$$

Similarly

$$P\left({}^B/_D\right) = \frac{0.02 \times 1/4}{0.025} = 0.2$$

$$P\left({}^C/_D\right) = \frac{0.04 \times 1/4}{0.025} = 0.4$$

$$P(D) = P\left({}^D/_A\right)P(A) + P\left({}^D/_B\right)P(B) + P\left({}^D/_C\right)P(C) = 0.0325$$